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WHEN IS THE UNION OF TWO UNIT INTERVALS A SELF-SIMILAR SET SATISFYING THE OPEN SET CONDITION?

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Abstract. Let U_λ be the union of two unit intervals with gap λ. We show that U_λ is a self-similar set satisfying the open set condition if and only if U_λ can tile an interval by finitely many of its affine copies (admitting different dilations). Furthermore each such λ can be characterized as the spectrum of an irreducible double word which represents a tiling pattern. Some further considerations of the set of all such λ's, as well as the corresponding tiling patterns, are given.

1. Introduction

Let $\{S_jx = c_jx + d_j\}_{j=1}^m$ be an iterated function system (IFS) on $\mathbb{R}$ such that $|c_j| < 1$ for all $1 \le j \le m$. Due to Hutchinson [4], there is a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^m S_j(K).$$

The set K is called the *self-similar set* generated by the IFS $\{S_j\}_{j=1}^m$. It is easy to analyze the geometric structure and calculate the dimensions of K when the IFS $\{S_j\}_{j=1}^m$ satisfies the so-called *open set condition* (OSC): there exists a non-empty bounded open set $U \subset \mathbb{R}$ such that

$$\bigcup_{j=1}^m S_j(U) \subset U$$

with the union disjoint. There are some equivalent definitions for the OSC (see, e.g. [10]). Nevertheless, it seems that there is no generic finite algorithm to determine whether a given IFS satisfies the OSC. Without confusion, we will say that a self-similar set satisfies the open set condition if it is generated by an IFS satisfying

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the open set condition. In particular, we say that a set E is a *SSOSC* if E is a self-similar set satisfying the open set condition.

It arises a much natural and fundamental problem that when a classic geometric object is a SSOSC. To our best knowledge, so far this problem has not been addressed and studied. In this paper we just consider some very special and simple cases. For $\lambda \geq 0$, let $U_\lambda = [0, 1] \cup [1 + \lambda, 2 + \lambda]$ denote the union of two unit intervals with gap $\lambda \geq 0$. We would like to know for which parameter λ, U_λ is a SSOSC. The question, to our surprise, is significantly more complicated than what we had anticipated. In this paper we present some partial answers.

To state our results, we first introduce some notations. For any integer $n \geq 2$, a word $\mathbf{w} = w_1 \cdots w_{2n}$ over some alphabet is called an *n-letter double word* if each letter in $\mathbf{w}$ appears exactly twice. Especially let Ω_n denote the set of all n-letter double words over the alphabet $\{1, \ldots, n\}$. For each word $\mathbf{w} \in \Omega_n$, we associate it with an $n \times n$ matrix $M_{\mathbf{w}} = (m_{i,j})_{1 \leq i,j \leq n}$ in the following way:

$$m_{i,j} = \begin{cases} 0, & \text{if } i = j, \\ \text{occurence of } j \text{ between the two letters } i\text{'s in } \mathbf{w}, & \text{otherwise.} \end{cases} \tag{1.1}$$

We call $M_{\mathbf{w}}$ the *incidence matrix of* $\mathbf{w}$. For example, let $\mathbf{w} = 123213$. Then

$$M_{\mathbf{w}} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

For each $\mathbf{w} \in \Omega_n$, we denote $\rho(\mathbf{w}) := \rho(M_{\mathbf{w}})$ the spectral radius of $M_{\mathbf{w}}$. Since $M_{\mathbf{w}}$ is non-negative, $\rho(\mathbf{w})$ is just the largest eigenvalue of $M_{\mathbf{w}}$ (see, e.g., [5]). Without confusion we just call $\rho(\mathbf{w})$ the *spectrum* of $\mathbf{w}$. For any finite double word $\mathbf{w}$ (not necessary over the alphabet $\{1, \ldots, n\}$) we can still define the incidence matrix and the spectrum $\rho(\mathbf{w})$ in the similar way. For each $n \in \mathbb{N}$, set

$$\Lambda_n = \{\rho(\mathbf{w}) : \ \mathbf{w} \in \Omega_n\}.$$

Our first result can be formulated as follows:

Theorem 1.1. *For $\lambda \in [0, +\infty)$, let $U_\lambda = [0, 1] \cup [1+\lambda, 2+\lambda]$. Then U_λ is a SSOSC if and only if $\lambda \in \Lambda := \bigcup_{n \geq 1} \Lambda_n$.*

We should point out that for each $\lambda \geq 0$, U_λ is always a self-similar set, not necessary to be a SSOSC. To see it, we may assume $\lambda > 0$ since the case $\lambda = 0$ is trivial. Choose an integer $m \geq \lambda$. Then the set $\bigcup_{i=0}^{m}(U_\lambda + i)$ is just the interval $[0, 2 + m + \lambda]$, where $U_\lambda + i$ denotes the set $\{x + i : \ x \in U_\lambda\}$. It follows that

$[0,1] = \bigcup_{j=1}^{m+1} \phi_j(U_\lambda)$, where $\phi_j(x) = \frac{x+j-1}{2+m+\lambda}$. Hence U_λ satisfies the self-similar relationship $U_\lambda = \bigcup_{j=1}^{2m+2} \phi_j(U_\lambda)$, where $\phi_{m+1+j}(x) = \phi_j(x)+1+\lambda$ for $1 \le j \le m+1$.

Theorem 1.1 gives a necessary and sufficient characterization for those λ such that the corresponding U_λ are SSOSC. For each n, the set Λ_n can be determined by a finite algorithm since Ω_n has only finitely many elements. Furthermore we have $\Lambda_n \subset \Lambda_{n+1}$. To see it, for any word $\mathbf{w} \in \Omega_n$, let $\mathbf{w}'$ be the word obtaining from $\mathbf{w}$ by adding the letters $n+1$ to the upper-most left-hand side and the upper-most righthand side of $\mathbf{w}$ respectively. By a simple observation of their incidence matrices we see that these two words have the same spectrum, i.e., $\rho(\mathbf{w}') = \rho(\mathbf{w})$. As we know, any element in Λ is the spectral radius of a non-negative integral matrix. Thus it must be a non-negative algebraic integer not less than its conjugates in modulus (see, e.g. [5]). However for a given such algebraic number, we have not yet found a finite algorithm to determine whether or not it belongs to Λ. In fact we have little understanding about the structure of the set Λ except the following result.

Theorem 1.2. *The set $\Lambda \cap [0,2]$ consists exactly of $0,1,2$ and the spectra of the words 123132, 12313424 and $\mathbf{w}_n = w_1 \cdots w_{2n} (n = 3, 4, \cdots)$ defined by $w_1 = w_3 = 1$, $w_{2n-2} = w_{2n} = n$ and $w_{2j-2} = w_{2j+1} = j$ for $2 \le j \le n-1$.*

In the following table we list the spectra of the words 123132, 12313424 and $\mathbf{w}_n$ for the first few n.

$\mathbf{w}$	Characteristic polynomial of $M_{\mathbf{w}}$	$\rho(\mathbf{w})(\approx)$
123132	$x^3 - 2x - 2$	1.769292
12313424	$x^4 - 3x^2 - 2x - 1$	1.919442
$\mathbf{w}_3$	$x^3 - 2x$	$\sqrt{2} \approx 1.414213$
$\mathbf{w}_4$	$x^4 - 3x^2 + 1$	$(1+\sqrt{5})/2 \approx 1.618033$
$\mathbf{w}_5$	$x^5 - 4x^3 + 3x$	$\sqrt{3} \approx 1.732050$
$\mathbf{w}_6$	$x^6 - 5x^4 + 6x^2 - 1$	1.801937
$\mathbf{w}_7$	$x^7 - 6x^5 + 10x^3 - 4x$	1.847759
$\mathbf{w}_8$	$x^8 - 7x^6 + 15x^4 - 10x^2 + 1$	1.879385

TABLE 1

It is quite natural to ask whether all Pisot numbers (i.e., algebraic integers greater than 1 whose algebraic conjugates are all less than 1 in modulus) will be among the spectra Λ. According to the above table, the answer is negative. Since $\rho(\mathbf{w}_n)$ is increasing (see Lemma 3.3), we can see, for example, that the smallest Pisot number (the largest root of $x^3 - x - 1$ which is approximately 1.3247..., see [2]) is not in Λ. Surely there are many other Pisot numbers are not in Λ, since the set of Pisot numbers has a cluster point at the golden ratio $\sqrt{5}+1)/2$ (see, e.g. [2]), which however is an isolated point in Λ.

We point out that our problem is related to the tiling theory. In fact, $\lambda \in \Lambda$ if and only if there exist finitely many affine maps ψ_i $(i = 1, \ldots, k)$ such that $\bigcup_{i=1}^k \psi_i(U_\lambda)$ is a non-empty interval and $\psi_i(\mathrm{int}(\mathrm{U}_\lambda))$ are disjoint (see Proposition 2.1). In another word, $\lambda \in \Lambda$ if and only if U_λ can "tile" an interval by using finitely many of its affine copies. In the above setting we admit dilations, which is different from the usual sense of tiling (see, e.g. [7, 8]). We will see that for some parameters λ, U_λ may tile an interval by several essentially distinct ways.

The paper is formulated as follows. We prove Theorem 1.1 in §2 and Theorem 1.2 in §3. In §4 we give some examples and present some unsolved questions.

2. The proof of Theorem 1.1

We first give several propositions.

Proposition 2.1. *Let K be the union of finitely many closed intervals. Then K is a SSOSC if and only if K can tile an interval by using finitely many of its affine copies, i.e., there exist finitely many affine maps ψ_i $(i = 1, \ldots, k)$ such that $\bigcup_{i=1}^k \psi_i(K)$ is a non-empty interval and $\psi_i(\mathrm{int}(\mathrm{K}))$ are disjoint.*

Proof. We first prove the "if" part. Assume that there exists a family of affine maps $\{\phi_i\}_{i=1}^k$ with $k \geq 2$ such that $\phi_i(\mathrm{int}(K))$ are disjoint and the union $U = \bigcup_{i=1}^k \phi_i(K)$ is an interval. Since K itself is the union of several disjoint intervals, there exists affine maps g_j $(j = 1, \ldots, m)$ such that

$$K = \bigcup_{j=1}^m g_j(U),$$

where $g_j(U)$ are disjoint. Hence we have

$$K = \bigcup_{j=1}^m \bigcup_{i=1}^k g_j \circ \phi_i(K),$$

where $g_j \circ \phi_i(K)$ are disjoint. It implies that K is a SSOSC.

Now we prove the "only if " part. To avoid the trivial case we assume that K is the union of at least two disjoint intervals and K satisfies the self-similar relation

$$K = \bigcup_{i=1}^{m} S_i(K),$$

where $S_i(\mathrm{int(K)})$ are disjoint. Take a large integer ℓ such that for each index $i_1 \cdots i_\ell$, the diameter of

$$S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_\ell}(K)$$

is less than the smallest gap between the intervals in K. This guarantees that each affine copy $S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_\ell}(K)$ is contained in one of the intervals in K. Let U be an interval in K. Since

$$K = \bigcup_{i_1 i_2 \cdots i_\ell} S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_\ell}(K),$$

we have

$$U = \bigcup_{i_1 i_2 \cdots i_\ell :\, S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_\ell}(K) \subset U} S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_\ell}(K).$$

This shows that K can tile U by its affine copies. □

For $k \geq 2$, let $K = K(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_{k-1})$ be the union of k ordered closed intervals of lengths α_i $(i = 1, \ldots, k)$ and gaps β_i $(i = 1, \ldots, k-1)$ respectively, with $\inf(K) = 0$. That is,

$$K(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_{k-1}) = \bigcup_{i=1}^{k} \left[\sum_{j=0}^{i-1} (\alpha_j + \beta_j), \alpha_i + \sum_{j=0}^{i-1} (\alpha_j + \beta_j) \right]$$

with convention $\alpha_0 = \beta_0 = 0$. By definition, we have just $U_\lambda = K(1, 1; \lambda)$.

Proposition 2.2. *Let $k \geq 2$. Suppose that $K = K(1, \ldots, 1; \lambda_1, \ldots, \lambda_{k-1})$ is the union of k intervals of unit length and gaps $\lambda_1, \ldots, \lambda_{k-1}$. Furthermore assume that K is a SSOSC. Then all the numbers λ_i $(1 \leq i \leq k)$ are Perron or Lind numbers (i.e., positive algebraic integers whose conjugates are no greater in moduli).*

Proof. Without loss of generality we show that λ_1 is an algebraic integer. To see it we let Δ be the first gap interval of K. Since K is a SSOSC, by Proposition 2.1, there exists a family of affine maps $\{\phi_i(x) = \rho_i x + d_i\}_{i=1}^m$ such that $\phi_i(K)$ are disjoint and their union is an interval. Therefore for each $1 \leq i \leq m$, the interval $\phi_i(\Delta)$ can be filled by some intervals of the form $\phi_j(U_\ell)$ where $j \neq i$ and U_ℓ is among the unit intervals in K. By comparing the lengths of these intervals, we obtain

$$|\rho_i| \lambda_1 = \sum_{j=1}^{m} t_{ij} |\rho_j|, \qquad i = 1, \ldots, m, \tag{2.2}$$

where t_{ij} is the number of distinct unit intervals U_ℓ such that $\phi_j(U_\ell)$ are contained in $\phi_i(\Delta)$. Therefore λ_1 is an eigenvalue of the $m \times m$ non-negative integral matrix (t_{ij}) associated with an eigenvector having positive entries. Hence λ_1 is the spectral radius of (t_{ij}). Due to Lind [9], λ_1 is a Perron number (i.e., a positive algebraic integer whose conjugates are smaller in moduli) or a Lind number (i.e., a positive algebraic integer η whose conjugates $|\eta'| \leq \eta$ and at least one $|\eta'| = \eta$). $\square$

We remark that Perron numbers were introduced by Lind in [9] and Lind numbers were introduced by Lagarias in [6]. Furthermore the result of the above proposition still remains valid if K is the union of several intervals of integral lengths.

Proof of Theorem 1.1.

Assume that $U_\lambda = K(1,1;\lambda)$ is a SSOSC. Then by Proposition 2.1, there exists a family of affine maps $\{\phi_i(x) = \rho_i x + d_i\}_{i=1}^m$ such that $\phi_i(U_\lambda)$ are disjoint and their union is an interval W. Denote by I_1 and I_2 the two unit intervals in U_λ, and by Δ the gap interval. Then the intervals $\phi_i(I_\ell)$ ($i = 1, \ldots, m$, $\ell = 1, 2$) tile W. The order of these intervals (from left to the right) induces an m-letter double word $\mathbf{w} = w_1 w_2 \cdots w_{2m}$, where $w_k = i$ if the k-th interval is $\phi_i(I_1)$ or $\phi_i(I_2)$. Using the identical argument in the proof of Proposition 2.2, we have

$$|\rho_i|\lambda = \sum_{j=1}^{m} t_{ij}|\rho_j|, \qquad i = 1, \ldots, m, \tag{2.3}$$

where t_{ij} is number of ℓ such that $\phi_j(I_\ell) \subset \phi_i(\Delta)$. According to the construction of $\mathbf{w}$, the matrix (t_{ij}) is just the incidence matrix $M_{\mathbf{w}}$ of $\mathbf{w}$. Therefore $\lambda \in \Lambda_m$.

Now we turn to the proof of the reverse part of the theorem. Assume that $\lambda \in \Lambda_m$ for some $m \in \mathbb{N}$. Then λ is the spectral radius of the incidence matrix $M_{\mathbf{w}}$ of some m-letter double word $\mathbf{w} = w_1 \cdots w_{2m}$. We assume that $\lambda > 0$, otherwise there is nothing to prove. Since $M_{\mathbf{w}}$ is non-negative, $M_{\mathbf{w}}$ has a nonnegative eigenvector $(\rho_1, \ldots, \rho_m)$ corresponding to λ (see, e.g., [5, Theorem 8.3.1].) Now we construct a family of intervals $\{I_i^{(j)} : 1 \leq i \leq m, 1 \leq j \leq 2\}$ in the following inductive way. Let J_1 be the interval (or point) of length ρ_{w_1} with left endpoint 0. Label J_1 as $I_{w_1}^{(1)}$. Assume J_k has been constructed well. Then J_{k+1} is the interval of length $\rho_{w_{k+1}}$ adjacent to J_k from the right hand side. Label J_{k+1} as $I_{w_{k+1}}^{(j)}$, where $j = 2$ if there is a $s < k+1$ such that $w_s = w_{k+1}$, and $j = 1$ otherwise. The above construct guarantees that for each i with $\rho_i \neq 0$, the union $\bigcup_{j=1}^2 I_{w_i}^{(j)}$ is an affine copy of U_λ. Hence the affine copies of U_λ can tile an interval. By proposition 2.2, U_λ is a SSOSC. $\square$

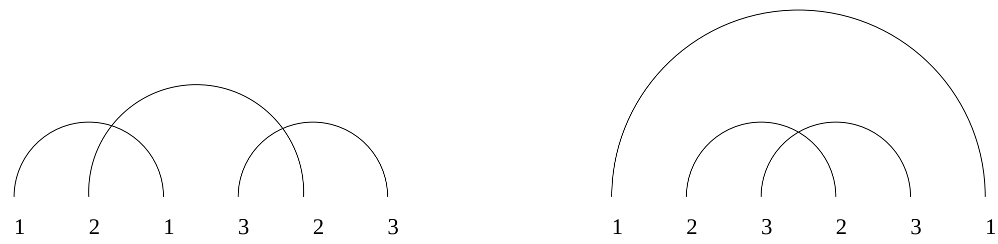

FIGURE 1. The graph representations of 121323 and 123231

Remark 2.3. *As an analogue, we may also give a necessary and sufficient condition for the parameter* λ *such that the set* $K = K(m, n, \lambda)$ *is SSOSC when* $m, n \in \mathbb{N}$ *are given.*

3. The proof of Theorem 1.2

We first introduce some definitions. A double word $\mathbf{w}$ is said to be *reducible* if there exists a strict sub-word $\mathbf{w}'$ of $\mathbf{w}$ such that $\mathbf{w}'$ is a double word. Conversely, $\mathbf{w}$ is said to be *irreducible* if it is not reducible. For instance, the word 123231 is reducible whilst 123213 is irreducible.

There is an intuitive way to determine the irreducibility of a given double word. To see it, let $\mathbf{w} = w_1 \cdots w_{2n}$ be a double word. We distribute $2n$ points on the real line and label them from left to right by the ordered letters in $\mathbf{w}$. Then we connect each pair of points, which are labeled by the same letter, by a half circle. Then, the union of this collection of half circles, denoted as $\Gamma(\mathbf{w})$, is called the *graph representation* of $\mathbf{w}$. In Figure 1, we give the graph representations of the words 121323 and 123231 respectively.

It is rather easy to see that

Lemma 3.1. *A double word* $\mathbf{w}$ *is irreducible if and only if its graph representation* $\Gamma(\mathbf{w})$ *is connected.*

We have another equivalent way to describe the irreducibility of a double word. Recall that a $n \times n$ non-negative matrix $A = (a_{ij})$ is called *irreducible* if for any $i, j \in \{1, \ldots, n\}$, there exists a word $i_1 i_2 \cdots i_n$ over $\{1, \ldots, n\}$ such that $i = i_1, j = i_n$ and $a_{i_k i_{k+1}} > 0$ for all $1 \leq k \leq n-1$.

Lemma 3.2. *A double word* $\mathbf{w}$ *is irreducible if and only if its incidence matrix is irreducible.*

Proof. Without loss of generality we assume that $\mathbf{w} \in \Omega_n$. First assume that $\mathbf{w}$ is irreducible. By Lemma 3.1, the graph representation $\Gamma(\mathbf{w})$ is connected. For $i = 1, \ldots, n$, let γ_i denote the half-circles connecting the letters i. Then for each pair of index i, j, there exists a connected path $\gamma_{i_1}, \ldots, \gamma_{i_n}$ from γ_i to γ_j. That is, $i_1 = i$, $i_n = j$ and γ_{i_k} intersects $\gamma_{i_{k+1}}$ for all $1 \leq k \leq n-1$. Observe that γ_{i_k} intersecting $\gamma_{i_{k+1}}$ implies that in the word $\mathbf{w}$ there is a letter i_{k+1} appears in the two letters i_k, and thus $m_{i_k i_{k+1}} > 0$. Therefore $M_{\mathbf{w}}$ is irreducible.

Now assume that $\mathbf{w}$ is reducible. Then $\mathbf{w}$ has a strict sub-word $\mathbf{w}'$ which is double. Let T denotes the set of letters appearing in $\mathbf{w}'$. Then by the definition of $M_{\mathbf{w}}$, for any $i \in T$ and $j \in \{1, \ldots, n\} \backslash T$, $m_{i,j} = 0$. This implies that $M_{\mathbf{w}}$ is reducible. To see it, take $i \in T$ and $j \in \{1, \ldots, n\} \backslash T$. Let $i_1 \cdots i_n$ be an arbitrary word over $\{1, \ldots, n\}$ with $i_1 = i$ and $i_n = j$. Let k be the smallest integer such that $i_k \in \{1, \ldots, n\} \backslash T$. Then $k > 1$ and $i_{k-1} \in T$. Thus $m_{i_{k-1} i_k} = 0$. Therefore $M_{\mathbf{w}}$ is reducible. This finishes the proof of the lemma. $\square$

The following lemma is needed in our proof of Theorem 1.2.

Lemma 3.3. *Let $\mathbf{w}$ be an irreducible double word in Ω_n $(n \geq 2)$. Then there exists $j \in \{1, \ldots, n\}$ such that the word $\mathbf{w}'$, obtained by removing the two letters j from $\mathbf{w}$, is still irreducible. Furthermore, $\rho(\mathbf{w}) > \rho(\mathbf{w}')$.*

Proof. Let $\Gamma(\mathbf{w})$ be the graph representation of $\mathbf{w}$. By Lemma 3.1, $\Gamma(\mathbf{w})$ is connected. Then there is a half-circle γ in $\Gamma(\mathbf{w})$ such that removing γ from $\Gamma(\mathbf{w})$, we get another connected graph representation, which implies the first part of the lemma. This fact comes from a more general result in graph theory, namely given any connected graph one can always remove a vertex and all edges connecting it so that the remaining graph is also connected (for a proof, see e.g. [1, Theorem 3.1.10]). To prove the second part, one observes that $M_{\mathbf{w}'}$ is just obtained from $M_{\mathbf{w}}$ by removing the entries of the j-th column and the j-th row of $M_{\mathbf{w}}$. Since by Lemma 3.2 $M_{\mathbf{w}}$ is irreducible, we have $\rho(M_{\mathbf{w}}) > \rho(M_{\mathbf{w}'})$ (see, e.g., [5, Exer 15, p 515]).

$\square$

Proposition 3.4. *$\lambda \in \Lambda$ if and only if λ is the spectrum of some finite irreducible double word.*

Proof. The "if" part is trivial. We only need to show the "only if" part. Assume that $\lambda \in \Lambda$. If $\lambda = 0$, then λ is the spectrum of the word 11 which is irreducible. Now assume $\lambda > 0$. By Theorem 1.1 and Proposition 2.1, U_λ can tile an interval by its affine copies. Let k be the smallest integer such that there are affine maps ψ_i $(i = 1, \ldots, k)$ such that $\bigcup_{i=1}^{k} \psi_i(U_\lambda)$ is a non-empty interval W and $\psi_i(\mathrm{int(U_\lambda)})$ are disjoint. Let I_1, I_2 be the two unit intervals in U_λ. Then W is just tiled by the intervals $\psi_i(I_j)$, $i = 1, \ldots, k$, $j = 1, 2$. Labeling each interval $\psi_i(I_j)$ in W by the letter i, we obtain a double word $\mathbf{w} \in \Omega_k$ by the natural order (from left to right) of $\psi_i(I_j)$ appearing in W. By the minimality of k, $\mathbf{w}$ is irreducible. Since if $\mathbf{w}$ has a strict sub-word $\mathbf{w}'$ which is double, then the corresponding intervals $\psi_i(I_j)$ ($j = 1, 2$, and i appears in $\mathbf{w}'$) will tile an interval, with a smaller number of affine copies of U_λ than k, which leads to a contradiction. By (2.3), λ is just the spectrum of $\mathbf{w}$. This finishes the proof. □

Let Ω'_n denote the set of all irreducible double words over $\{1, \ldots, n\}$ and set

$$\Lambda'_n = \{\rho(\mathbf{w}) : \ \mathbf{w} \in \Omega'_n\}.$$

By Proposition 3.4, $\Lambda = \bigcup_{n=1}^{\infty} \Lambda'_n$. To describe the structure of Ω'_n and Λ'_n, we first introduce some definitions. An operator $T : \Omega'_n \to \Omega'_n$ is called a *permutation operator* if there is a permutation σ on $\{1, \ldots, n\}$ such that $T(w_1 \cdots w_{2n}) = \sigma(w_1) \cdots \sigma(w_{2n})$ for any $\mathbf{w} = w_1 \cdots w_{2n} \in \Omega'_n$. Similarly T is called a *reflection* if $T(w_1 w_2 \cdots w_{2n}) = w_{2n} w_{2n-1} \cdots w_1$. Furthermore an operator T is called an *elementary operator* if it is the composition of finitely many the above two kind of operators. Two words $\mathbf{w}, \mathbf{w}' \in \Omega'_n$ are said to be *equivalent* and denoted by $\mathbf{w}' \sim \mathbf{w}$ if there is an elementary operator T such that $\mathbf{w}' = T(\mathbf{w})$. It is easy to see that $\mathbf{w} \sim \mathbf{w}'$ if and only if their graph representations are either the same or differ by a mirror reflection. It is trivial to see that the only element in $\Omega'_1/\sim$ is the word 11 which has the spectrum 0, and the only element in $\Omega'_2/\sim$ is the word 1212 which has the spectrum 1. In the following two tables we list all representative elements in $\Omega'_n/\sim$ for $n = 3, 4$, together with their characteristic polynomials and the numerical estimations of the spectra.

$\mathbf{w}$	Characteristic polynomial of $M_{\mathbf{w}}$	$\rho(\mathbf{w})(\approx)$
121323	$x(x^2 - 2)$	1.414214
123123	$(x - 2)(x + 1)^2$	2.000000
123132	$x^3 - 2x - 2$	1.769292

TABLE 2. Elements in $\Omega'_3/\sim$

Lemma 3.5. *Let $\mathbf{w}_n$ $(n \geq 3)$ be the words defined in Theorem 1.2. Then $\rho(\mathbf{w}_n) < 2$ and $\lim_{n\to\infty} \rho(\mathbf{w}_n) = 2$.*

$\mathbf{w}$	Characteristic polynomial of $M_{\mathbf{w}}$	$\rho(\mathbf{w})(\approx)$
12132434	$(x^2-x-1)(x^2+x-1)$	1.618034
12134234	$(1+x)(x^3-x^2-3x+1)$	2.170086
12134243	$x(x-2)(1+x)^2$	2.000000
12134324	x^4-3x^2-2x+1	1.940393
12314234	$x(1+x)(x^2-x-4)$	2.561553
12314243	$(x^2-2x-1)(1+x)^2$	2.414214
12314324	$x(x^3-4x-4)$	2.382976
12314342	x^4-3x^2-1-2x	2.052300
12324143	$(x^2+x+1)(x^2-x-3)$	2.302776
12341234	$(x-3)(1+x)^3$	3.000000
12341243	$(1+x)(x^3-x^2-4x-4)$	2.875130
12341324	$(1+x)(x^3-x^2-4x-4)$	2.875130
12341342	$(1+x)(x^3-x^2-3x-3)$	2.598675
12341423	$(1+x)(x^3-x^2-3x-3)$	2.598675
12341432	$(1+x)(x^3-x^2-2x-4)$	2.467504
12342143	$(x^2+2x+2)(x^2-2x-2)$	2.732051
12342413	x^4-3x^2-4x-1	2.234023

TABLE 3. Elements in $\Omega_4'/\sim$

Proof. It is easy to observe that for each $n \geq 3$, the sum of entries in the j-th row of $M_{\mathbf{w}_n}$is equal to 2 for $2 \leq j \leq n-1$, and 1 for $j = 1, n$. Since $M_{\mathbf{w}_n}$ is irreducible, by Gerschgorin's disk theorem, $\rho(\mathbf{w}_n) < 2$. Since $\mathbf{w}_n$ is obtained by removing the two letters $(n+1)$ from $\mathbf{w}_{n+1}$, by Lemma 3.3, we have $\rho(\mathbf{w}_{n+1}) > \rho(\mathbf{w}_n)$. Therefore, the limit

$$a = \lim_{n\to\infty} \rho(\mathbf{w}_n)$$

exists and $1 < a \leq 2$. In the following we show that $a = 2$.

Let $f_n(t) = \det(tI_n - M_{\mathbf{w}_n})$ be the characteristic polynomial of the matrix $M_{\mathbf{w}_n}$. It is easy to show by mathematical induction that the sequence $\{f_n\}$ satisfies the following recurrence relation: $f_{n+2}(t) = tf_{n+1}(t) - f_n(t)$ for $n \geq 3$. Taking $t = a$ and $u_n = f_n(a)$, we have

$$u_{n+2} = au_{n+1} - u_n, \qquad n \geq 3. \tag{3.4}$$

Since $\rho(\mathbf{w}_n)$ is the largest positive root of the monoid polynomial f_n and $\rho(\mathbf{w}_n) < a$, we have $u_n = f_n(a) > 0$. Assume that a is strictly smaller than 2. Then the polynomial $x^2 = ax - 1$ has only two imaginary roots. By a well-known result (see , e.g., [3]), there are infinitely many terms in $\{u_n\}$ taking negative values, which leads to a contradiction. $\square$

Proof of Theorem 1.2. By Proposition 3.4 and the above tables, it suffices to prove the following claim:

Claim: For any $n \geq 5$, the unique element in $\Omega'_n/\sim$ whose spectrum less than 2 is the word $\mathbf{w}_n$ defined in Theorem 1.2.

To prove the claim, we first verify the claim directly for the case $n = 5$ by estimating the spectra of elements in $\Omega'_5/\sim$ by Matlab (there are exact 135 elements). Now we prove the claim for any $n \geq 5$ by mathematical induction. Assume that the claim is true for some $n \geq 5$. Let $\mathbf{u}_{n+1}$ be a word in $\Omega'_{n+1}/\sim$ such that $\rho(\mathbf{u}_{n+1}) < 2$. We show below that $\mathbf{u}_{n+1}$ is just equivalent to $\mathbf{w}_{n+1}$. To see this, by Lemma 3.3, $\mathbf{u}_{n+1}$ is equivalent to a word $\mathbf{u}'_{n+1}$ in Ω'_{n+1} which is obtained by adding two letters $n+1$ in a word $\mathbf{u}_n \in \Omega'_n$, and $\rho(\mathbf{u}_{n+1}) > \rho(\mathbf{u}_n)$. Hence $\rho(\mathbf{u}_n) < 2$ and thus $\mathbf{u}_n$ is equivalent to $\mathbf{w}_n$. Without loss of generality we may take $\mathbf{u}_n = \mathbf{w}_n$ and assume that $\mathbf{u}_{n+1}$ is an irreducible word obtained by adding two letters $n+1$ in $\mathbf{w}_n$. If the letters $n+1$ are just added on the two sides of the first letter 1 in $\mathbf{w}$ (or the last letter n) respectively, then $\mathbf{u}_{n+1}$ is just equivalent to $\mathbf{w}_{n+1}$. In other cases, we may either delete the two letters 1 (this is the case that none of the two letters $n+1$ lie between the two letters 1, or the first letter $n+1$ lies between two 1's and the second lies between two n's) or delete the two letters n from $\mathbf{u}_{n+1}$ (this is the case that none of the two letters $n+1$ lie between the two n's) to get an irreducible word $\mathbf{u}$ in Ω_n not equivalent to $\mathbf{w}_n$, which implies that $\rho(\mathbf{u}_{n+1}) > \rho(\mathbf{u}) > 2$ and contradicts the assumption $\rho(\mathbf{u}_{n+1}) < 2$. This finishes the proof of the claim. $\square$

4. Examples and unsolved questions

We first give an example to show that for some parameters λ, U_λ may tile an intervals in some essentially different ways.

Assume that U_λ can tile an interval by its affine copies $\psi_i(U_\lambda)$, $i = 1, \dots, n$. Furthermore assume that each strict subfamily of these copies can not tile an interval. As in the proof of Proposition 3.4, we can use an irreducible word $\mathbf{w}_n \in \Omega'_n$ to represent this tiling pattern. In the following example one can see that U_λ may have different tiling patterns.

Example 4.1.

- $\lambda = 2$. *U_λ has the tiling patterns* 123123*,* 12134243*,* 121345435*.*

- $\lambda = 3$. *U_λ has the tiling patterns* 12341234, 1231452453, 123241564635.
- $\lambda = \sqrt{2}+1$. *U_λ has the tiling patterns* 12314243, 1234254513, 121324565364.

In the table 4, we give all the elements in $\Omega'_6/\sim$ having $\rho_{\mathbf{w}} = 3$.

$\mathbf{w}$	Characteristic polynomial of $M_{\mathbf{w}}$
121342565346	$(x-1)(x-3)(x+1)^4$
121343564265	$x^2(x-3)(x+1)^3$
121345264653	$x^2(x-3)(x+1)^3$
121345426365	$(x-3)(x+1)(x^2+x-1)(1+x+x^2)$
121345436256	$x^2(x-3)(x+1)^3$
121345462563	$x^2(x-3)(x+1)^3$
123124563564	$(x-1)(x-3)(x+1)^4$
123124563645	$(x-1)(x-3)(x+1)^4$
123143526465	$(x-1)(x-3)(x+1)^4$
123143526546	$(x-1)(x-3)(x+1)^4$
123145356264	$x^2(x-3)(x+1)^3$
123241564635	$x^2(x-3)(x+1)^3$
123245364615	$(x-3)(x+1)(x^2+x-1)(x^2+x+1)$
123245365416	$x^2(x-3)(x+1)^3$
123413564652	$x^2(x-3)(x+1)^3$
123425465163	$x^2(x-3)(x+1)^3$

TABLE 4. Elements $\mathbf{w}$ in $\Omega'_6/\sim$ such that $\rho_{\mathbf{w}} = 3$

Now we give an example of a SSOSC which is the union of three unit intervals admitting non-integer gaps.

Example 4.2. *Let $K = K(1,1,1;1,2\lambda+1)$, where $\lambda \approx 2.5212$ is the largest root of the polynomial of x^3-4x-6. Then K is a SSOSC. To see it, let $K' = K\cup(K+1)$. Then $K' = K(4,2;2\lambda) = 2K(2,1;\lambda)$, which can tile an interval by the pattern* 123213.

In the following we present some unsolved problems:

- Is there a finite algorithm to determine if or not $\lambda \in \Lambda$ for any given λ?
- For each $\lambda \in \Lambda$, are there only finitely many words $\mathbf{w}$ in $\bigcup_n \Omega'_n$ such that $\rho(\mathbf{w}) = \lambda$? In another word, does U_λ have at most finitely many tiling patterns for any given λ? For instance, how about for $\lambda = 3$?

- Is Λ closed? if not, does the closure of Λ contain non-empty interior?
- Can we determine, for example, when a polygon in the plane is a SSOSC?

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